BALLS ARE MAXIMIZERS OF THE RIESZ-TYPE FUNCTIONALS WITH SUPERMODULAR INTEGRANDS

HICHEM HAJAIEJ

Abstract. For a large class of supermodular integrands, we establish conditions under which balls are the unique (up to translations) maximizers of the Riesz-type functionals with constraints.

1. Introduction

Over the last decades, one field of intense research activity has been the study of extremals of integral functionals. The Riesz-type kind has attracted growing attention and played a crucial role in the resolution of Choquard's conjecture in a breakthrough paper by E. H. Lieb [1]. The determination of cases of equality in the Riesz-rearrangement inequality has also received a large amount of interest from mathematicians due to its connection with many other functional inequalities and its several applications to physics [2, 3, 4]. Variational problems for steady axisymmetric vortex-rings in which kinetic energy is maximized subject to prescribed impulse involves Riesz-type functionals with constraints. In [5], G. R. Burton has proved the existence of maximizers in an extended constraint set, he has also showed that the maximizer is Schwarz symmetric (up to translations). His method hinges on a resolution of an optimization of a Riesz-type functional under constraint [5, Proposition 8]. The purpose of this paper is to answer the more general question: When do maximizers of the Riesz-type functional inherit the symmetry and monotonicity properties of the integrand involved in it?

The method of G. R. Burton [5] cannot apply to solve the above problem. In this paper, we develop a self-contained approach. Let us give here a foretaste of our ideas. First, we recall that:

A Riesz-type functional is a functional of the form:

$$R(f,g) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Psi\left(f(x), g(y)\right) r(x,y)\, dx\, dy.$$

In this paper, we will consider $r(x,y) = j\left(|x-y|\right)$. We are interested in the following maximization problem:

(P1) $$\sup_{(f,g)\in C} J(f,g)$$

where

$$J(f,g) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Psi\left(f(x), g(y)\right) j\left(|x-y|\right) dx\, dy. \tag{1.1}$$

and

$$C = (f,g) : \begin{cases} f : & \mathbb{R}^n \to \mathbb{R}; 0 \le f \le k_1 \text{ and } \int_{\mathbb{R}^n} f \le \ell_1 \\ g : & \mathbb{R}^n \to \mathbb{R}; 0 \le g \le k_2 \text{ and } \int_{\mathbb{R}^n} g \le \ell_2 \end{cases} \tag{1.2}$$

ℓ_1, k_1, ℓ_2, k_2 are positive numbers.

For supermodular operators Ψ and nonincreasing functions j, we know that $J(f,g) \leq J(f^*,g^*)$ [4, Theorem 1], where u^* denotes the Schwarz symmetrization of u. Hence the problem reduces to:

(P2) $$\sup_{(f^*,g^*)\in C} J\left(f^*,g^*\right)$$

For continuous integrands Ψ having the N-Luzin property (for any subset N having Lebesgue measure zero, $\Psi(\mathrm{N})$ has the same property), lemma 2.6 enables us to assert that (P2) is equivalent to an optimization of a Hardy-Littlewood type functionals where balls are maximizers. We will then extend this study to supermodular non-continuous bounded functions Ψ thanks to the decomposition of these functions into $\tilde{\Psi}\left(\varphi_1(s_1),\varphi_2(s_2)\right)$ in the spirit of [4, 6]. The approximation of unbounded supermodular functions by bounded ones inheriting the monotonicity properties will enable us to prove that balls are maximizers in the general case.

Main Result:

Let $\Psi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ be a H-Borel function satisfying:

(Ψ1) Ψ vanishes at hyperplanes;

(Ψ2) $\Psi(b,d) - \Psi(b,c) - \Psi(a,d) + \Psi(a,c) \geq 0$ for all $0 \leq a < b$ and $0 \leq c < d$;

(Ψ3)(i) $\Psi(tx,b_2) - t\Psi(x,b_2) - \Psi(tx,b_1) + t\Psi(x,b_1) \leq 0$ for all $x \geq 0$, $0 \leq b_1 < b_2$ and $0 < t < 1$;

(Ψ3)(ii) $\Psi(a_2,ty) - t\Psi(a_2,y) - \Psi(a_1,ty) + t\Psi(a_1,y) \leq 0$ for all $y \geq 0$, $0 \leq a_1 < a_2$ and $0 < t < 1$;

(j1) j is nonincreasing.

Suppose in addition that Ψ is continuous with respect to each variable and has the N-Luzin property, then for all $(f_1,f_2) \in C$

$$J\left(f_1,f_2\right) \leq J\left(k_1 1_{B_1}, k_2 1_{B_2}\right)$$

where B_1 and B_2 are centered in the origin, 1_B is the characteristic function of B, and $\mu(B_1) = \ell_1/k_1$, $\mu(B_2) = \ell_2/k_2$. Moreover, if (Ψ2) and (Ψ3) hold with strict inequality, j is strictly decreasing and $J\left(f_1,f_2\right) < \infty$ for any $(f_1,f_2) \in C$, then (P1) is attained by exactly two couples $(k_1 1_{B_1}, k_2 1_{B_2})$ and (h_1,h_2) where h_1 and h_2 are translates by the same vector of $k_1 1_{B_1}$ and $k_2 1_{B_2}$ (respectively).

2. Notations and preliminaries

Definition 2.1. *If $A \subset \mathbb{R}^n$ is a measurable set of finite Lebesgue measures μ, we define A^*, the symmetric rearrangement of the set A to be the open ball centered at the origin whose volume is that of A, thus $A^* = \{x \in \mathbb{R}^n : |x| < r\}$ with $V_n r^n = \mu(A)$, V_n is a constant.*

For a nonnegative measurable function u on $\mathbb{R}^n$, we require u to vanish at infinity in the sense that all its positive level sets $\{x \in \mathbb{R}^n : u(x) > t\}$ having finite measure for $t > 0$. The set of these functions is denoted by F_n. The symmetric decreasing rearrangement u^* of u is the unique upper semicontinuous, nonincreasing radial function that is equimeasurable with u. Explicitly, $u^*(x) = \int\limits_0^\infty \mathbf{1}^*_{\{u>t\}}(x)\,dt$ where $\mathbf{1}^*_A = \mathbf{1}_{A^*}$. We say that u is Schwarz symmetric if $u \equiv u^*$.

Definition 2.2. *A reflexion σ on $\mathbb{R}^n$ is an isometry with the properties:*

- (i) *$\sigma_x^2 = \sigma_x \circ \sigma_x = x$ for all $x \in \mathbb{R}^n$;*
- (ii) *the fixed point set of H_0 of σ separates $\mathbb{R}^n$ into two half spaces H_+ and H_- that are interchanged by σ;*

(iii) $|x - x'| < |x - \sigma_{x'}|$ *for all* $x, x' \in H_+$.
H_+ *is the half space containing the origin.*

The two point rearrangement or polarization of a real valued function u with respect to a reflection σ is defined by:

$$u^{\sigma_x} = \begin{cases} \max\{u(x), u(\sigma_x)\}, x \in H_+ \cup H_0, \\ \min\{u(x), u(\sigma_x)\}, x \in H_-. \end{cases} \tag{2.1}$$

Lemma 2.3. *Let* $j : [0, \infty) \to \mathbb{R}$ *be a nonincreasing function then* $\nu(x) = \int_{\mathbb{R}^n} j(|x-y|)\, h(y)\, dy$ *is radial and radially decreasing for any Schwarz symmetric function* h. *If in addition* j *is strictly radially decreasing then* ν *also inherits this property.*

Proof: we will use [7, Lemma 2.8]: $u = u^* \Leftrightarrow u = u^\sigma$ for all σ. It is sufficient to prove that $u(x) \geq u(\sigma_x)$ for all $x \in \mathbb{R}^n$, all σ.

$$\begin{aligned} u(x) &= \int_{H^+} j(|x-y|)\, h(y) + j(|x-\sigma_y|)\, h(\sigma_y)\, dy \\ u(\sigma_x) &= \int_{H^+} j(|\sigma_x-y|)\, h(y) + j(|\sigma_x-\sigma_y|)\, h(\sigma_y)\, dy \\ u(x) - u(\sigma_x) &= \int_{H_+} j(|x-y|)\,[h(y) - h(\sigma_y)] - j(\sigma_x - y)\,[h(y) - h(\sigma_y)]\, dy \\ &= \int_{H_+} (j(|x-y|) - j(\sigma_x - y))\,(h(y) - h(\sigma_y))\, dy \end{aligned}$$

By (iii) $|x-y| < |\sigma_x - y|$, it follows that $j(|x-y|) \geq j(|\sigma_x - y|)$. On the other hand h is Schwarz symmetric, hence $h(y) \geq h(\sigma_y)$ for all $y \in H_+$, the conclusion follows.

Definition 2.4. *Let* $\Psi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$*:*

(a) Ψ *is supermodular if* $(\Psi 2)$ *holds.*
(b) *We say that* Ψ *vanishes at hyperplanes if* $\Psi(s_1, 0) = \Psi(0, s_2) = 0$ *for all* $s_1, s_2 \geq 0$.

An important property of functions satisfying (c) is that the composition $(x, y) \mapsto \Psi(f(x), g(y))$ is measurable on $\mathbb{R}_+$ for every $f, g \in F_n$. Hence $j(|x-y|)\, \Psi(f(x), g(y))$ is measurable on $\mathbb{R}_+ \times \mathbb{R}_+$. In the spirit of [4] and [6], we obtain:

Lemma 2.5. *Assume that* $\Psi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ *is a supermodular bounded function vanishing at hyperplanes. Then there exist two bounded nondecreasing functions* φ_1 *and* φ_2 *on* $\mathbb{R}_+$ *with* $\varphi_i(0) = 0$ *and a Lipschitz continuous function* $\tilde{\Psi}$ *on* $\mathbb{R}^2_+$ *such that* $\Psi(u, v) = \tilde{\Psi}(\varphi(u), \varphi(v))$.

Proof: First, we will prove the following: If φ is a nondecreasing real-valued function defined on an interval I, then for every f on I satisfying $|f(u) - f(v)| < c(\varphi(v) - \varphi(u))$ where $u < v \in I$, c is a constant, there exists a Lipschitz continuous function $\tilde{f} : \mathbb{R} \to [\inf f, \sup f]$ such that $f(x) = \tilde{f} \circ \varphi(x)$ (2.0). If f is nondecreasing then $\tilde{f}$ is nondecreasing also.
The result is obvious for $t = \varphi(v)$ and $s = \varphi(u) < t$ since we have

$$|\tilde{f}(t) - \tilde{f}(s)| = |f(\varphi(v)) - f(\varphi(u))| \leq c(\varphi(v)) - (\varphi(u)) = c(t-s).$$

Now $\tilde{f}$ has a unique extension to the closure of the image and the complement consists of a countable number of disjoint bounded intervals, it is sufficient to interpolate $\tilde{f}$ linearly between the values, that were assigned to end-points. By construction $f = \tilde{f} \circ \varphi$ and $\tilde{f}(\mathbb{R}) = [\inf f, \sup f]$ the extension we have made by linear interpolation preserves of course the modulus of continuity of $\tilde{f}$: $|\tilde{f}(t) - \tilde{f}(s)| \leq c(t-s)$ for all $t > s$. If f is nondecreasing, it is easy to check that this property is inherited by $\tilde{f}$.
Now we can prove our lemma:
First note that the fact that Ψ is supermodular and vanishes at hyperplanes imply that it is nondecreasing with respect to each variable and it is nonnegative. Now set $\varphi_1(u) = \lim_{u\to+\infty} \Psi(u,v)$.
φ_1 is well-defined on $\mathbb{R}_+$ since Ψ is bounded and nondecreasing in the second variable. By the supermodularity of Ψ, it follows that

$$\Psi(u+h_1, v+h_2) - \Psi(u, v+h_2) - \Psi(u+h_1, v) + \Psi(u,v) \geq 0$$

for any u, v, h_1 and $h_2 \geq 0$.
Letting h_2 tend to infinity, we obtain

$$\varphi_1(u+h_1) - \varphi(u) \geq \Psi(u+h_1, v) - \Psi(u,v) \geq 0$$

for all $u, v, h_1 \geq 0$.
For a fixed v, the last inequality enables us to apply (2.0) to $\Psi(\cdot, v)$. Hence, there exists Ψ^1 such that: $\Psi(u,v) = \Psi^1(\varphi_1(u), v)$. A moment's consideration shows that Ψ^1 inherits all the properties of Ψ. Now set $\varphi_2(v) = \lim_{u\to+\infty} \Psi(u,v)$, a similar argument ensures us that there exists $\tilde{\Psi}$ such that $\Psi^1(\varphi_1(u), u) = \tilde{\Psi}^1(\varphi_1(u), \varphi_2(v))$.
$\tilde{\Psi}$ has the same monotonicity property as Ψ^1 and consequently as Ψ. Note that $\varphi_1(0) = \varphi_2(0) = 0$ and the monotonicity properties of Ψ imply that φ_1 and φ_2 are nondecreasing.

Lemma 2.6. *Let $l, k > 0$, $D = \{h : \mathbb{R}^n \to \mathbb{R} : 0 \leq h(x) \leq k$ and $\int_{\mathbb{R}^n} h(x)\,dx \leq l\}$. Suppose that $\Gamma : \mathbb{R}_+ \to \mathbb{R}$ is a function satisfying:*

1. *$\Gamma(0) = 0$,*
2. *$\Gamma(tx) \leq t\Gamma(x)$ for all $x \geq 0$ and $0 < t < 1$.*
   Assume also that
3. *$u : \mathbb{R}^n \to \mathbb{R}$ is a Schwarz symmetric function. Then for every $\nu \in D$: $\int_{\mathbb{R}^n} u(x)\Gamma(\nu(x))\,dx \leq \int_{\mathbb{R}^n} u(x)\Gamma(k\mathbf{1}_B(x))\,dx$ where B is the ball centered at the origin with $\mu(B) = \ell/k$.*

Proof: (2) implies that

$$\begin{aligned}\int_{\mathbb{R}^n} u(x)\Gamma(\nu(x))\,dx \leq \int_{\mathbb{R}^n} u(x)\Gamma(k)\frac{\nu(x)}{k}\,dx &= \Gamma(k)\left[\int_B u(x)\left[\frac{\nu(x)}{k} - 1 + 1\right]dx + \int_{\mathbb{R}^n - B} \frac{u(x)\nu(x)}{k}\,dx\right] \\ &= \int_{\mathbb{R}^n} u(x)\Gamma\left(k\mathbf{1}_{B(x)}\right)dx \\ &\quad +\Gamma(k)\left[\int_B u(x)\left[\frac{\nu(x)}{k} - 1\right]dx + \int_{\mathbb{R}^n - B} \frac{u(x)\nu(x)}{k}\,dx\right]dx.\end{aligned}$$

Using (3), it follows that the above integrals are $\leq \int_{\mathbb{R}^n} u(x)\Gamma\left(k\mathbf{1}_{B(x)}\right)dx + \Gamma(k)u(r)\left[\int_{\mathbb{R}^n} \frac{\nu(x)}{k}\,dx - \mu(B)\right]$ where $\mu(B) = V_r r^n$ (see definition 2.1). Thus $\int_{\mathbb{R}^n} u(x)\Gamma(\nu(x))\,dx \leq \int_{\mathbb{R}^n} u(x)\Gamma\left(k\mathbf{1}_{B(x)}\right)dx$, since $\int_{\mathbb{R}^n} \frac{\nu(x)}{k}\,dx \leq \mu(B) = \ell/k$.

If additionally $\int_{\mathbb{R}^n} u(x)\Gamma\left(\nu(x)\right)\,dx < \infty$ for any $\nu \in D$, (2) holds with strict inequality and u is strictly decreasing, we can prove that for every $\nu \in D$: $\int_{\mathbb{R}^n} u(x)\Gamma\left(\nu(x)\right)\,dx < \int_{\mathbb{R}^n} u(x)\Gamma\left(k\mathbf{1}_{B(x)}\right)\,dx$.

3. Proof of the result

For the convenience of the reader, the proof will be divided in three parts.
First part: We suppose that: $\Psi(\cdot, s_2)$ is absolutely continuous for every $s_2 \geq 0$, and $\Psi(s_1, \cdot)$ is absolutely continuous for every $s_1 \geq 0$.
First note that (Ψ1) and (Ψ2) imply that Ψ is a non-decreasing function with respect to each variable and it is nonnegative.
Let $(f_1, f_2) \in C$, (Ψ2) and (j1) imply that

$$\begin{aligned} J\left(f_1, f_2\right) \leq J\left(f_1^*, f_2^*\right) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Psi\left(f_1^*(x), f_2^*(y)\right) j\left(|x-y|\right)\,dx\,dy \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left(\int_0^{f_2^*(y)} F\left(f_1^*(x), s\right)\,ds\right) j\left(|x-y|\right)\,dx\,dy \end{aligned}$$

where $\Psi(s_1, s_2) = \int_0^{s_2} F\left(s_1, u\right)\,du$.
Applying Tonelli's theorem (see (3.0)), we obtain:

$$J\left(f_1^*, f_2^*\right) = \int_0^{\infty}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} j\left(|x-y|\right)\mathbf{1}_{\{y\in\mathbb{R}^n : f_2^*(y)\geq s\}} F\left(f_1^*(x), s\right)\,dy\,dx\,ds.$$

Setting $u(x,s) = \int_{\mathbb{R}^n}\mathbf{1}_{\{y\in\mathbb{R}^n : f_2^*(y)\geq s\}} j\left(|x-y|\right)\,dy$, it follows from lemma 2.3 that u is radial and radially decreasing with respect to x for every fixed s.

$$J\left(f_1^*, f_2^*\right) = \int_0^{\infty}\int_{\mathbb{R}^n} u(x,s) F\left(f_1^*(x), s\right)\,dx\,ds.$$

Now for a fixed $s_1 \geq 0$, $\Psi(s_1, x_2) - \Psi(s_1, x_1) = \int_{x_1}^{x_2} F(s_1, t)\,dt \geq 0$ for $x_2 \geq x_1$; from which we deduce that $F(s_1, t)$ is nonnegative for almost every $t \geq 0$. (3.0)
On the other hand, $0 = \Psi(0, s_2) = \int_0^{s_2} F(0, u)\,du$. By the nonnegativity of F, we conclude that $F(0, s) = 0$ for almost every $s \geq 0$.
Moreover (Ψ3) says that: $\Psi(tx, d) - t\Psi(x, d) - \Psi(tx, c) + t\Psi(x, c) \leq 0$ for every $x \geq 0$, $d \geq c \geq 0$.
Integrating this inequality, we have $\int_c^d F(tx, u) - tF(x, u)\,du \geq 0$ for every $x \geq 0$; $d \geq c \geq 0$.
Hence $F(tx, u) \leq tF(x, u)$ for all $x \geq 0$, $t \in]0, 1[$ and almost every $u \geq 0$.
This shows that for almost every $s \geq 0$, the function $u(x,s)F\left(f_1^*(x), s\right)$ satisfies all the hypotheses of lemma 2.6, consequently:
For almost every $s \geq 0$ $\int_{\mathbb{R}^n} u(x,s)F\left(f_1^*(x), s\right)\,dx \leq \int_{\mathbb{R}^n} u(x,s)F\left(k_1\mathbf{1}_{B_1}(x), s\right)\,dx$ and

$$J\left(f_1^*, f_2^*\right) \leq J\left(k_1\mathbf{1}_{B_1}, f_2^*\right). \tag{3.1}$$

Using the same argument, we easily conclude that

$$J\left(k_1\mathbf{1}_{B_1}, f_2^*\right) \leq J\left(k_1\mathbf{1}_{B_1}, k_2\mathbf{1}_{B_2}\right). \tag{3.2}$$

By [4, Theorem 2] we know that:

$$J\left(f_1, f_2\right) \leq J\left(f_1^*, f_2^*\right). \tag{3.3}$$

Combining these three inequalities, we obtain:

$$J(f_1,f_2) \leq J(f_1^*,f_2^*) \leq J(k_1\mathbf{1}_{B_1},f_2^*) \leq J(k_1\mathbf{1}_{B_1},k_2\mathbf{1}_{B_2}).$$

If in addition, we have strict inequality in (Ψ2) and (Ψ3), j is strictly decreasing and $J(f_1,f_2)<\infty$ for any $f_1,f_2 \in C$ then [4, Theorem 2] asserts that equality occurs in (3.3) if and only if there exists $x_0 \in \mathbb{R}^n$ such that $f_1 = f_1^*(\cdot - x_0)$ and $f_2 = f_2^*(\cdot - x_0)$.
On the other hand, by lemma 2.6, equality occurs in (3.1) if and only if $f_1^* = k_1\mathbf{1}_{B_1}$. Similarly equality holds in (3.2) if and only if $f_2^* = k_2\mathbf{1}_{B_2}$.
Conclusion: we have proved that for any absolutely continuous function Ψ satisfying (Ψ1), (Ψ2), (Ψ3) with a kernel j satisfying (j1) $(k_1\mathbf{1}_{B_1},k_2\mathbf{1}_{B_2})$ is a maximizer of J under the constraint C. If additionally (Ψ2), (Ψ3) hold with strict inequality j is strictly decreasing and $J(f_1,f_2)<\infty$ for all $(f_1,f_2)\in C$ then $(k_1\mathbf{1}_{B_1},k_2\mathbf{1}_{B_2})$ is the unique maximizer of (P1) (up to a translation).
Remark 1: Ψ is a nondecreasing function with respect to each variable, it is then of bounded variations. The absolute continuity is then equivalent to its continuity and the fact that it satisfies the N-Luzin property.
Remark 2: We can remove condition (Ψ1) from our theorem by modifying (Ψ3) and adding an integrability assumption in a same way as [8, Proposition 3.2].
Part 2: Ψ is bounded.
Applying lemma 2.5, we know that there exist φ_1,φ_2 such that $\Psi(s_1,s_2) = \tilde{\Psi}(\varphi_1(s_1),\varphi_2(s_2))$, where $\tilde{\Psi}$ is Lipschitz continuous with respect to each variable, there exist a function $\tilde{F}$ defined on $\mathbb{R}_+$ such that $\tilde{\Psi}(s_1,s_2) = \int_0^{s_2} \tilde{F}(s_1,u)\,du$.

$$\begin{aligned} J(f_1,f_2) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \tilde{\Psi}(\varphi_1(f_1^*(x)),\varphi_2(f_2^*(x)))\, j(|x-y|)\,dx\,dy \\ &= \int_0^\infty \left(\int_{\mathbb{R}^n} \nu(x,s)\tilde{F}(\varphi_1(f_1^*(x)),s)\right) dx\,ds \end{aligned}$$

where $\nu(x,s) = \int_{\mathbb{R}^n} \mathbf{1}_{\{y\in\mathbb{R}^n:\varphi_2(f_2^*(y))\geq s\}} j(|x-y|)\,dy$. The function $\mathbf{1}_{\{y\in\mathbb{R}^n:\varphi_2(f_2^*(y))\geq s\}}$ is Schwarz-symmetric for every s since φ_2 is nondecreasing. We can then apply Part 1 and the result follows.
Remark 3: Here we cannot obtain a uniqueness result since φ_1 and φ_2 do not inherit the strict monotonicity properties of Ψ.
Part 3: Ψ is not bounded.
For $L>0$, set $\Psi^L(s_1,s_2) = \Psi(\min(s_1,L),\min(s_2,L))$. It is easy to check that Ψ^L inherits all the properties of Ψ stated in our result. Moreover Part 2 applies to Ψ^L since it is a bounded function. Noticing that $\Psi^L \to \Psi$, the monotone convergence theorem enables us to conclude.

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Justus-Liebig-Universität Giessen, Mathematisches Institut, Arnd Str 2, 35392 Giessen, Germany
E-mail address: hichem.hajaiej@gmail.com